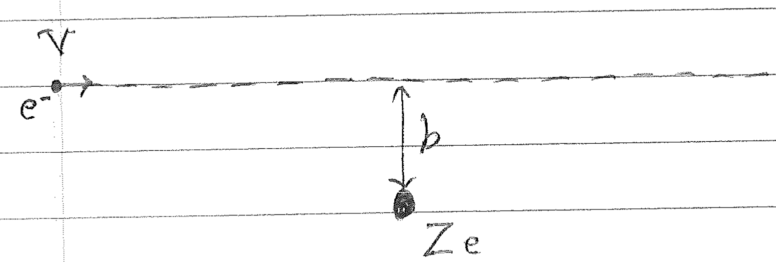
Lec 9: 09/19/2018

Bremsstrahlung:

As mentioned before, conservation of energy and momentum does not allow an isolated charged particle to spontaneously absorb or emit radiation. This conforms with the fact that only an accelerating charged particle can radiate. Emitting photons by a charged particle is possible when another source of energy and momentum lies nearby. This could be the Coulomb field an ion. Under the guise of electron-proton scattering, this is one of the most common manifestations of "Bremsstrahlung" radiation in astrophysics.

Consider an electron that has velocity v moving in the Coulomb field of a static ion:

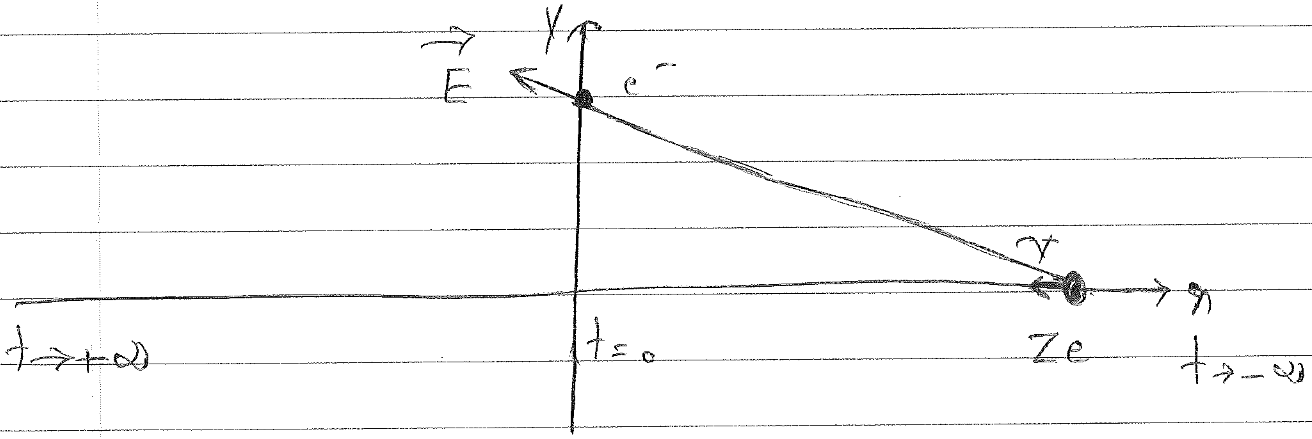


The force from the Coulomb field accelerates the electron. For non-relativistic motion, $v \ll c$, we can use the relation for angle-integrated distribution of radiation from the electron in the electric dipole approximation:

$$\frac{dW}{d\omega} = \frac{8\pi e^2}{3c^3} |\vec{a}_{(\omega)}|^2$$

It is convenient to move to the rest frame of the electron. We note that $\vec{a}$ is the same in both the lab frame and the electron rest frame in the non-relativistic limit. In the case of relativistic motion, we need to perform a Lorentz transformation back to the lab frame to find $\frac{dW}{d\omega}$ there.

In the electron rest frame, we have the following picture:



$$E_{\parallel} = \frac{Ze\gamma v t}{(b^2+\gamma^2 v^2 t^2)^{3/2}} \quad , \quad E_{\perp} = \frac{\gamma Z e b}{(b^2+\gamma^2 v^2 t^2)^{3/2}}$$

Here $\parallel$ and $\perp$ denote the components of the electric field that are parallel and perpendicular to the direction of the ion's motion respectively (i.e., along the x and y axes respectively).

Thus:

$$a_{\parallel}(t) = \frac{-\gamma Z e^2 v t}{m_e (b^2+\gamma^2 v^2 t^2)^{3/2}} \quad , \quad a_{\perp}(t) = \frac{-\gamma Z e^2 b}{m_e (b^2+\gamma^2 v^2 t^2)^{3/2}}$$

Here, we assume that the electron does not move significantly during the ion's motion. In particular, its displacement in the perpendicular direction is less than the impact factor b.

We note that the magnetic field induced by the ion can be neglected because we are in the non-relativistic regime.

After making a Fourier transformation to the frequency domain, we have:

$$a_{\parallel}(\omega) = \frac{-1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma Z e^2 v t}{m_e (b^2 + \gamma^2 v^2 t^2)^{3/2}} e^{i\omega t}\, dt$$

$$a_{\perp}(\omega) = \frac{-1}{2\pi}\int_{-\infty}^{+\infty} \frac{\gamma Z e^2 b}{m_e (b^2 + \gamma^2 v^2 t^2)^{3/2}} e^{i\omega t}\, dt$$

These integrals can be written in closed form:

$$a_{\parallel}(\omega) = \frac{-1}{2\pi}\frac{Ze^2}{m_e}\frac{1}{\gamma b v}\left[2iyK_0(y)\right]$$

$$a_{\perp}(\omega) = \frac{-1}{2\pi}\frac{Ze^2}{m_e}\frac{1}{b v}\left[2yK_1(y)\right]$$

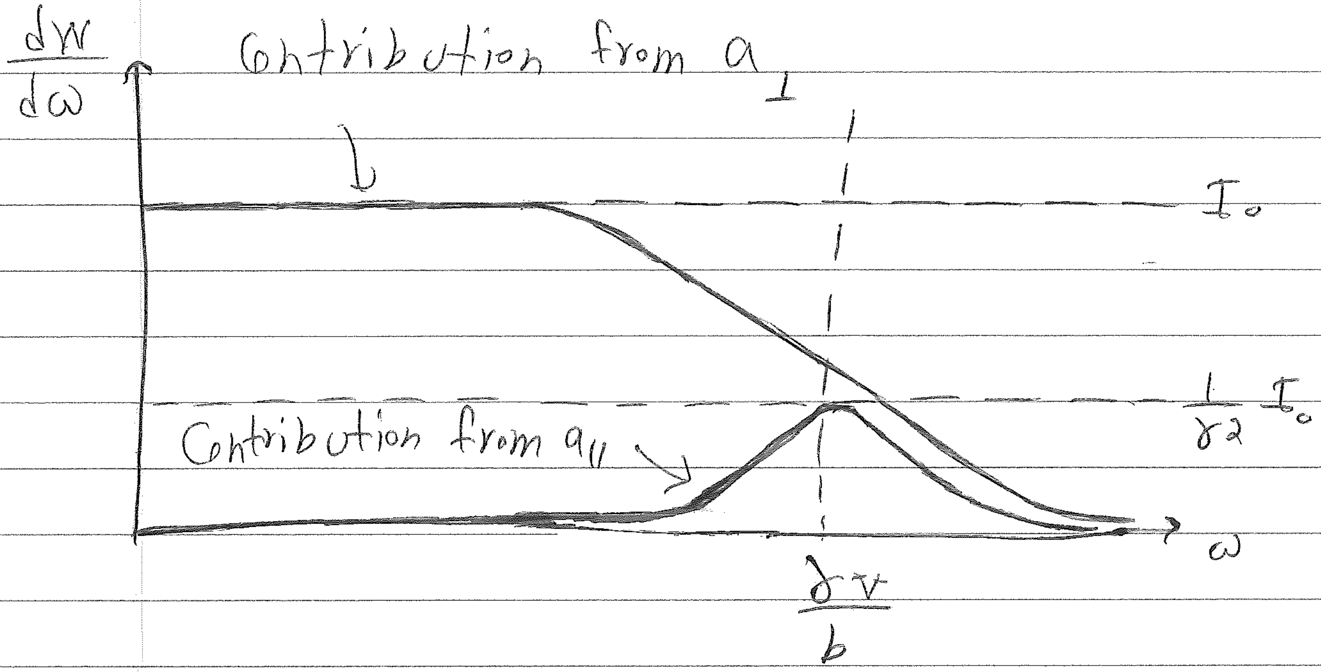
Here $y \equiv \frac{\omega b}{\gamma v}$, and K_0, K_1 are modified Bessel functions of order zero and one respectively.

Putting things together, we find:

$$\frac{dW}{d\omega} = \frac{8\pi e^2}{3c^3}\left[\,|a_{\parallel}(\omega)|^2 + |a_{\perp}(\omega)|^2\right] \Rightarrow \frac{dW}{d\omega} = \frac{8Z^2e^6}{3\pi c^3 m_e^2}$$

$$\frac{\omega^2}{\gamma^2 v^2}\left[\frac{1}{\gamma^2}K_0^2\left(\frac{\omega b}{\gamma v}\right) + K_1^2\left(\frac{\omega b}{\gamma v}\right)\right]$$

It is interesting to plot the spectrum, displaying the terms arising from $a_{\parallel}$ and $a_{\perp}$ separately:



We see that in the relativistic regime, $\gamma \gg 1$, the contribution of $a_\parallel$ is negligible. Moreover, both contributions drop exponentially at frequencies higher than $\sim \frac{\gamma v}{b}$. This cut-off can be understood from the fact that the duration of interaction is roughly $\tau = \frac{2b}{\gamma v}$ in the electron rest frame. Therefore the spread in the frequency domain is expected to be $\Delta\omega \sim \frac{2\pi}{\tau} \sim \frac{\gamma v}{b}$. The exponential cut-off implies that there is little power emitted at frequencies greater than $\frac{\gamma v}{b}$.

It is also instructive to study the asymptotic limits of $\frac{dW}{d\omega}$. At high frequencies, we have:

$$\frac{dW}{d\omega} \simeq \frac{4Z^2e^6}{3c^3m_e^2}\frac{1}{\gamma v^3}\left[\frac{1}{\gamma^2}+1\right]\exp\left(\frac{-2\omega b}{\gamma v}\right) \qquad \omega \gg \frac{\gamma v}{b}$$

At low frequencies, on the other hand, we have:

$$\frac{dW}{d\omega} \simeq \frac{8Z^2e^6}{3\pi c^3m_e^2}\frac{1}{b^2v^2}\left[1-\frac{1}{\gamma^2}\left(\frac{\omega b}{\gamma v}\right)^2\ln^2\left(\frac{\omega b}{\gamma v}\right)\right] \qquad \omega \ll \frac{\gamma v}{b}$$

In the low frequency limit, the second term inside the bracket can be neglected. The spectrum therefore asymptotes to a constant, which is evident from the plot on the previous page. This can be understood as follows. As far as low frequencies are concerned, the momentum impulse felt by the electron is a delta function since the duration of interaction is much shorter than the period of these modes. As a result, the spectrum is flat at these frequencies.

Finally, we have to integrate over all relevant impact parameters. The number density of ions in the electron rest frame is γn, where n is the density in the lab frame. The number of encounters per unit time is $\gamma n v$, which results in:

$$\frac{d^2 W}{d\omega dt} = \int_{b_{min}}^{b_{max}} 2\pi b \gamma n v \frac{dW}{d\omega} db$$

Focusing on frequencies lower than $\frac{\gamma v}{b}$, where $\frac{dW}{d\omega}$ is significant, we find:

$$\frac{d^2 W}{d\omega dt} \simeq \frac{16 Z^2 e^6}{3c^3 m_e^2} \frac{\gamma n}{v} \ln\left(\frac{b_{max}}{b_{min}}\right)$$

Here, b_{min} and b_{max} denote the minimum and maximum values of the impact parameter, respectively, for which the assumptions we made in deriving $\frac{dW}{d\omega}$ are valid.

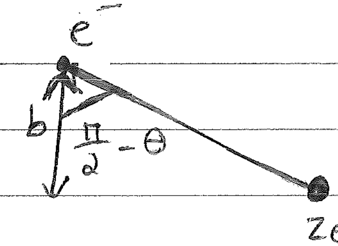
b_{max} can be estimated as follows. The interaction is the strongest when the ion is at the distance of closest approach to the electron. This results in a characteristic time scale $\frac{2b}{\gamma v}$. For a given frequency ω, we need to have $\omega \lesssim \frac{\gamma v}{2b}$. Therefore,

$$b_{max} \sim \frac{\gamma v}{\omega}$$

As for b_{min}, there is a classical and a quantum restriction.

Classically, the momentum impulse on the electron by the time the ion reaches its distance of closest approach is found to be:

$$P = \int_{-\infty}^{0} F_{\perp}\, dt = -\frac{Ze^2}{bv} \int_{0}^{\frac{\pi}{2}} \sin\theta \, d\theta = \frac{Ze^2}{bv}$$



The average velocity of the electron is:

$$\bar{v} = \frac{P}{2m_e} = \frac{Ze^2}{2bv\, m_e}$$

The distance travelled by the electron within time $\frac{\tau}{2} = \frac{b}{\gamma v}$ will then be $\frac{Ze^2}{2\gamma v^2 m_e}$, which needs to be less than b. Thus,

$$b_{min}\ (\text{classical}) \sim \frac{Ze^2}{2\gamma m_e v^2}$$

However, the electron cannot be localized within a distance smaller than its de Broglie wavelength $\frac{\hbar}{p}$. This results in:

$$b_{min}(\text{quantum}) \sim \frac{\hbar}{2m_e v} \qquad (\Delta x \Delta p \geq \frac{\hbar}{2})$$

We then have:

$$b_{min} = \max\left[\frac{Ze^2}{2\gamma m_e v^2}, \frac{\hbar}{2m_e v}\right]$$

We note that:

$$\frac{b_{min}(\text{quantum})}{b_{min}(\text{classical})} = \frac{1}{Z}\frac{1}{\alpha}\frac{v}{c} \qquad (\gamma \simeq 1 \text{ in the non-relativistic limit})$$

Here $\alpha \equiv \frac{e^2}{\hbar c}$ is the fine structure constant where $\alpha \simeq \frac{1}{137}$.

This implies that at high velocities, $\frac{v}{c} \gtrsim \alpha$, we should use $b_{min}(\text{quantum})$, while at low velocities, $\frac{v}{c} < \alpha$, we need to use $b_{min}(\text{classical})$.

With proper choices of b_{max} and b_{min}, as discussed above, we can calculate $\frac{d^2W}{d\omega\, dt}$. Then, by integrating over ω, we can find the energy loss rate of the electron due to radiation.